The variational solution of electric and magnetic circuits

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Voltage and current distributions in electric circuits comply with a stationary power condition. An alternative circuit analysis technique can be derived from this property. The unknowns can be varied until the power consumption of the circuit is calculated to be unchanging. At this point the voltages and currents are at their true solution. This variational approach to circuit analysis has been generally overlooked. No doubt this is due to the sufficiency of conventional circuit analysis techniques. However, a closer examination of the variational approach reveals a straightforward analytic procedure with unifying properties and graphically illustrated solutions. These features provide new educational opportunities and insight. As a research tool the variational approach offers a unified method of solving problems which include fields and devices. The application to nonlinear circuits is also of current interest. An introductory step-by-step guide to the variational procedure for electric and magnetic networks is described in this article.

Introduction

As an engineering analysis tool, the variational approach is well established and has a solid foundation in both physics and mathematics. It is universally used with numerical techniques and is accredited for its versatility, physical interpretation and elegance of formation. With variational methods the unique solution of an engineering problem is found via the stationary point of a suitable energy or power expression. Starting with a postulated solution the behaviour of energy is examined as the postulated solution is varied. An invariant energy quantity, or stationary point, indicates a true solution. In the early days, this was done graphically; now, computers can do these numerical experiments very efficiently and accurately.

Variational methods are extended here to examine the behaviour of electric and magnetic circuits. In comparison to conventional circuit analysis methods the stationary power technique offers a simplified formulation, a method of solution for nonlinear devices and relation to a universal principle. This approach is also consistent with the variational/finite-element method widely used for distributed field problems. From an educational viewpoint the stationary solution of circuits provides a very descriptive introductory example to variational methods. The use of differential calculus and graphical illustrations also provides new and useful teaching material.

A variational power equation for circuits can be constructed from the instantaneous power of an electrical network. Either unknown nodal voltage values or branch currents are used as the unknown variables. As power is a scalar quantity there are no restrictions on current and voltage polarities. Each circuit branch element is represented by an entry into the characteristic power equation. With nonlinear devices the voltage/current characteristic has to be integrated to form a power expression. One equation describes the entire circuit and its stationary point coincides with the circuit's true solution. The formulation lends itself to approximation techniques or can be solved rigorously using matrix methods.

The stationary power procedure for lumped element circuits is described here by having recourse to some simple steady-state circuits with DC and AC sources.
Characteristic power equations are formed for linear capacitive, resistive and reactive impedance circuits. A formal analytic procedure is developed and supplemented by graphical solutions. A nonlinear resistor problem is treated before considering a nonlinear magnetic circuit. The approach described can be extended to evaluate harmonics in nonlinear circuits and to produce the unified solution of lumped-element/distributed-field problems.

Minimum energy solution of capacitor circuits

Theoretical studies in electrical engineering can either involve a distributed space or lumped-element circuits. In distributed problems electromagnetic fields and potentials are calculated. A simple class of distributed problem is that of electrostatics, where electric charge is time stationary. The electric-field and voltage solution of a capacitor plate arrangement is one example. It is well known that in electrostatics the voltage solution in a distributed space corresponds to a minimum stored energy. Indeed this result is routinely used in numerical techniques which rely on variational methods. In variational methods the unknowns of the problem are varied until the calculated value of a suitable energy expression is unchanging.

Electric lumped-element capacitor circuits also exhibit this minimum energy behaviour. Energy analysis in circuits is straightforward and therefore provides an excellent introduction to variational energy methods. Consider the capacitor circuit illustrated in Fig. 1. The potentials \( V_1 \) and \( V_2 \) are unknown. If \( V_1 \) and \( V_2 \) are varied then the overall stored energy of the circuit will also vary. In this process there is a unique solution pair of \( V_1 \) and \( V_2 \) which minimises the circuit's stored energy. The first step is to construct an energy characteristic equation for the circuit. In electromagnetics this is called a functional. The energy stored by a capacitor, \( C \), is a function of its differential plate voltage, \( V' \), and is written as \( \frac{1}{2} CV'^2 \). In the circuit of Fig. 1 there are five capacitors and there will be subsequently five terms in the characteristic equation. Taking each capacitor in turn this gives

\[
P(V_1, V_2) = \frac{1}{2} V_1 (V_1 - 100)^2 + \frac{1}{2} V_2 (V_2 - 100)^2 + \frac{1}{2} V_3 (V_3 - 100)^2 + \frac{1}{2} V_4 (V_4 - 100)^2 + \frac{1}{2} V_5 (V_5 - 100)^2
\]

If \( V_1 \) and \( V_2 \) are treated as trial functions and varied between 0 and 100 volts then \( P(V_1, V_2) \) exhibits a minimum value when \( V_1 = 22.24 \) volts and \( V_2 = 26.76 \) volts. The energy surface can also be plotted and is illustrated in Fig. 2.

Fig. 2. Variation of the voltage \( V_1 \) yields a minimum energy solution for the capacitor circuit of Fig. 1.

To find the stationary point of eqn. 1. differentiating eqn. 1 with respect to \( V_1 \) and \( V_2 \) and set each result to zero. The two equations generated are

\[
\frac{\partial P(V_1, V_2)}{\partial V_1} = (V_1 - 100) + 3(V_1 - V_2) + 4V_1 = 0
\]

\[
\frac{\partial P(V_1, V_2)}{\partial V_2} = (V_2 - 100) + 3(V_2 - V_1) = 0
\]
and these simultaneous equations have the solution given above. It is also instructive to examine the nature of the turning point using calculus. Forming the second order derivatives of eqn. 1 yields

\[
\frac{\partial^2 P(V_1, V_2)}{\partial V_1^2} = 8 \\
\frac{\partial^2 P(V_1, V_2)}{\partial V_2^2} = 10
\]

which are both positive quantities and indicative of minimum turning points as illustrated in Fig. 2.

An advantage of demonstrating variational methods using circuits is that the result is easily corroborated using conventional circuit analysis techniques. The circuit of Fig. 1 can be solved in at least three other ways. Thévenin's theorem in conjunction with circuit reduction techniques is one approach. This method would however not work for a more complicated circuit arrangement. Another approach involves substituting an AC supply for the battery, using capacitive reactances, and taking the solution as the limit of zero frequency. A less obvious method would be to use the equation of continuity which describes the indestructibility of charge. Each approach identically generates the equations 2 and 3.

**Variational solution of a resistor circuit**

Purely resistive circuits also comply with the stationary power principle. The power dissipated in each resistor branch of a network can either be written in terms of unknown nodal voltages or unknown branch currents. If node voltages are used then all sources must be converted to voltage sources. If branch currents are used then current supplies are necessary. A resistor circuit with voltage supplies is illustrated in Fig. 3. Conventional methods such as nodal analysis or mesh currents would quickly yield the solution of \( V_1 \) equal to 5.18 V and \( V_2 \) equal to 6.44 V. From an educational viewpoint the variational solution of this circuit has some distinct advantages. Firstly it utilizes a universal energy principle used throughout science and engineering. Secondly, as scalar power terms are used in the formulation, initial errors with current and voltage polarities are avoided. Thirdly, one characteristic equation describes the entire circuit and for low-order problems it can be graphically illustrated. Finally, the stationary-power approach utilizes elementary calculus and provides an excellent application of these techniques.

If the terminal voltage of a resistor, \( R \), is \( V \) then one way of calculating the power dissipated in the element

\[
\frac{\partial P(V_1, V_2)}{\partial V_1} = 2(V_2 - 100) - 2(V_1 - V_2) + 5V_2 = 0 \quad (3)
\]

is to substitute an AC supply for the battery, using capacitive reactances, and taking the solution as the limit of zero frequency. A less obvious method would be to use the equation of continuity which describes the indestructibility of charge. Each approach identically generates the equations 2 and 3.
is to form the equation $V^2/R$. Using this equation and taking each branch resistance in turn, the power characteristic equation $P(V_1, V_2)$ of the circuit can be constructed. There are five branch resistances, therefore there are five contributing terms to this equation:

$$P(V_1, V_2) = \frac{(V_1 - 8)^2}{2} + \frac{V_1^2}{6} + \frac{(V_2 - 12)^2}{8} + \frac{(V_2 - 4)^2}{6} + \frac{(V_2 - 4)^2}{8}$$

(6)

The voltages $V_1$ and $V_2$ can be varied as trial functions and the power can be calculated for each point. This variational process is illustrated in Fig. 4, where a minimum is illustrated at the true solution given earlier. Alternatively, the minimum power point can be found using differential calculus. Differentiate $P(V_1, V_2)$ with respect to the unknowns $V_1$ and $V_2$ and set each result to zero to find the stationary point. The two simultaneous equations generated are as follows:

$$\frac{\partial P(V_1, V_2)}{\partial V_1} = 2(V_1 - 8) + \frac{V_1^2}{6} + 2(V_1 - 4) = 0 \quad (7)$$

$$\frac{\partial P(V_1, V_2)}{\partial V_2} = -2(V_1 - 8) + 2(V_2 - 12) + \frac{(V_2 - 4)^2}{8} = 0 \quad (8)$$

These equations are exactly those which are produced when using conventional nodal analysis and have a solution of $V_1$ equal to 5.18 V and $V_2$ equal to 6.44 V. The second order derivatives of eqn. 6 yield positive quantities which imply a minimum turning point, as illustrated in Fig. 4.

### Variational solution of an AC circuit

Electrical circuits with AC signal sources and reactive elements are described in terms of impedances rather than pure resistances. The unknown nodal voltages or branch currents in the circuit become complex quantities describing both magnitude and phase. A suitable variational expression for this class of problem is deduced from the apparent power quantity which can be written in terms of the adjacent unknown nodal voltages. The characteristic equation for the circuit is constructed from the linear addition of apparent power consumed by each branch. There are three impedance branches in the circuit of Fig. 5 and therefore there are three entries in the characteristic equation. The result is written in terms of $V_1$ as

$$P(V_1) = \frac{(V_1 - 10)^2}{\beta} + \frac{V_1^2}{4} + \frac{V_2^2}{8}$$

(9)

The real and imaginary parts of the complex variable $V_1$ may be varied and the value of apparent power calculated. Fig. 6 illustrates this construction in the vicinity of the stationary point. This point coincides with the true solution for $V_1$. It is of note that the complex functional yields a stationary saddle point. Analytically the stationary point is determined from the first variation of eqn. 9. This yields the following equation:

$$\frac{dP(V_1)}{dV_1} = 2(V_1 - 10) \beta^3 + 2V_1 \beta^2 + V_1 \beta = 0 \quad (10)$$

which has the correct solution of $V_1 = 12.3+j18.5$. 

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**Fig. 5** An AC source circuit with reactive impedances and one unknown complex nodal voltage $V_i$. 

**Fig. 6** For the AC circuit of Fig. 5 the variation of the real and imaginary parts of $V_1$ yields a stationary saddle point.
Variational solution of a nonlinear resistor circuit

Nonlinear passive components which are characterised by a single-valued current/voltage relationship also obey the stationary power principle. The contribution of each element in the nonlinear circuit to the characteristic power equation is evaluated using integral terms. For example, the power dissipated in a nonlinear resistor connected between nodes $V_1$ and $V_2$ is

$$P = \int_{V_1}^{V_2} IV \, dV$$

(11)

Once the characteristic power equation has been formed, the variational solution procedure is identical to that for linear DC circuits.

For the circuit in Fig. 7, the characteristic power equation formed using eqn. 11 is

$$P = \frac{(18 - V)^2}{12} + \frac{\sqrt{21V^2}}{3}$$

(12)

The stationary point of this equation gives the solution. Differentiating with respect to $V$ and setting the results to zero gives

$$\frac{dP}{dV} = \frac{V}{6} + \sqrt{\frac{V}{2}} - 3 = 0$$

(13)

Thus the solution is $V = 6.87$ V. The graph of power as a function of $V$ in Fig. 7 illustrates the occurrence of a minimum at the solution point.

Variational solution of a nonlinear magnetic circuit

Magnetic circuits are inherently nonlinear due to the saturation and hysteresis properties of magnetic materials such as mild steel. Such a circuit is schematically illustrated in Fig. 8. The core is of mild steel and has a magnetisation curve detailed in Fig. 9. The magnetic field intensity, $H$, is a nonlinear function of the magnetic flux density, $B$. The requirement is to find the flux density in the air-gap using a variational minimum power approach. The analysis presumes that the flux path is uniform in the core and air-gap. The power dissipated, $P$, in a magnetic circuit can be written in terms of the magnetomotive force, MMF, and the reluctance, $R$, of the section:

$$P = \frac{\text{MMF}^2}{R}$$

(14)

The MMF generated, $Nl$, and the MMF dropped in the air-gap, $Hd$, can be used to define the MMF drop in the magnetic core, $Hd$, as

$$Hd = NI - H_a$$

(15)

where $l$ and $a$ are the mean path lengths of the core and air-gap, respectively. The total power dissipated then becomes

$$P = \frac{(NI - H_a)^2}{R} + \frac{(H_a)^2}{R_e}$$

(16)
where the reluctances of the core, $R_c$, and gap, $R_g$, are defined using the cross-sectional area, $A$, and flux density, $B$, as

$$R_c = \frac{\mu H_c}{A}$$  \hspace{1cm} (17)$$

$$R_g = \frac{\mu H_g}{A}$$  \hspace{1cm} (18)$$

Substituting eqns. 17 and 18 into 16 and using the relationship $E = \frac{bH}{R}$ in the air-gap, the power dissipated in the magnetic circuit becomes

$$P = BA \left[ \frac{(NI - B\frac{\mu}{\mu_c})^2}{UH} + \frac{R_g}{\mu_c} \right]$$  \hspace{1cm} (19)$$

Eqn. 19 has two unknowns, $H$ and $B$, which are also related by the magnetisation curve of Fig. 9. Substituting an $(H, B)$ pair from Fig. 9 into eqn. 19 yields a scalar power quantity. This quantity varies with different $(H, B)$ pairs. It exhibits a minimum at the true solution. This variational process is illustrated in Fig. 10. A minimum quantity is clearly exhibited at a flux density of around 1.35 tesla. Conventional methods can be used to check this result by working backwards from the solution to the MMF eqn. 15.

**Conclusion**

Variational methods do not consist of solving a set of equations but in approximating to a trajectory or functional. Physically interpreted the method displaces a system from its dynamic equilibrium position and examines the displaced behaviour. In the application to electrical circuits the behaviour of a power functional is examined at the unknown voltages or currents are displaced. When the first variation is set to zero the unknowns are in the vicinity of their correct solution. This approach has been described by examining the behaviour of some linear and nonlinear electric and magnetic circuits.

There are a number of new possibilities which arise from the application of variational methods to circuits. A unified electromagnetic field solver with a lumped-element device facility would be a useful proposition for applications which combine irregular geometries with resistors, capacitors etc. Additional information can be generated concerning the nature of a solution and its convergence. Confidence limits on convergence parameters may be of particular use in nonlinear applications.

**References**

1. MIKHUN, S. G.: "Variational methods in mathematical physics" (Macmillan, New York, 1964)

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