Regularization, renormalization, and dimensional analysis: Dimensional regularization meets freshman E&M

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We illustrate the dimensional regularization technique using a simple example from electrostatics. This example illustrates the virtues of dimensional regularization without the complications of a full quantum field theory calculation. We contrast the dimensional regularization approach with the cutoff regularization approach, and demonstrate that dimensional regularization preserves translational symmetry. We then introduce minimal subtraction and modified minimal subtraction schemes to renormalize the result. Finally, we consider dimensional transmutation as encountered in the case of compact "extra dimensions."
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I. DIMENSIONAL REGULARIZATION

A. Introduction and motivation

In 1999, Gerardus 't Hooft and Martinus J. G. Veltman received the Nobel Prize in Physics[1,2] "for elucidating the quantum structure of electroweak interactions in physics." In particular, they demonstrated that the non-Abelian electroweak theory could be consistently renormalized to yield unique and precise predictions. A key ingredient of their demonstration was the development of the dimensional regularization technique.[3–5] That is, instead of working in precisely $D=4$ space-time dimensions, they generalized the dimension to be a continuous variable so they could calculate the theory in $D=4.01$ or $D=3.99$ dimensions.

An important property of the dimensional regularization is that it respects gauge and Lorentz symmetries,[6] in contrast to the other regularization schemes such as cutoff schemes which violate these symmetries. The symmetries of the electroweak theory play a critical role in determining the dynamics of the particles and their interactions. Because it respects these symmetries, dimensional regularization has become an essential tool for the calculation of field theories.

Although dimensional regularization is a powerful and elegant technique, most examples and applications of dimensional regularization are in the context of complex higher-order quantum field theory calculations involving gauge and Lorentz symmetries. However, the virtues of dimensional regularization can be exhibited without the distractions of the associated quantum field theory complexities.

In the present paper we will apply the dimensional regularization method to the electric potential of an infinite line of charge.[7,8] The example is simple enough for undergraduates to understand, yet contains many of the concepts we encounter in a quantum field theory calculation. We will contrast the symmetry-preserving dimensional regularization approach with a symmetry-violating cutoff approach.

Imagining a variable number of dimensions can be a productive exercise. To explain the weak nature of the gravitational force physicists have recently posited the existence of "extra dimensions." After considering space-time dimensions in the neighborhood of $D=4$, we briefly contemplate wider excursions of $D=4,5,6,\ldots$ dimensions.

II. DIMENSION ANALYSIS: THE PYTHAGOREAN THEOREM

To illustrate the utility of dimensional regularization and dimensional analysis, we warm-up with an example. Our goal is to demonstrate the Pythagorean theorem, and our method will be dimensional analysis.

Consider the right triangle displayed in Fig. 1(a). From the angle-side-angle theorem, this triangle can be uniquely specified using the angles $\{\theta,\phi\}$ and the hypotenuse c. To construct a formula for the area of the triangle, A_c, using only the variables $\{c,\theta,\phi\}$, we note that c has dimensions of length, and $\{\theta,\phi\}$ are dimensionless. From dimensional analysis, the area of the triangle must have dimensions of length squared. Because c is the only dimensional quantity, the formula for A_c must be of the form

$$A_c = c^2 f(\theta,\phi), \tag{1}$$

where $f(\theta,\phi)$ is an unknown dimensionless function. Note that $f(\theta,\phi)$ cannot depend on the length c because this dependence would spoil the dimensionless nature of $f(\theta,\phi)$.

We observe that we can divide the original triangle of Fig. 1(a) into two similar triangles of hypotenuses a and b, as shown in Fig. 1(b). Again, by using the angle-side-angle theorem, we can represent the area of these triangles, A_a and A_b, in terms of the variables $\{a,\theta,\phi\}$ and $\{b,\theta,\phi\}$, respectively. From dimensional considerations, these areas must be proportional to a^2 and b^2. Thus, we obtain

$$A_a + A_b = a^2 f(\theta,\phi) + b^2 f(\theta,\phi). \tag{2}$$

Because all three triangles are similar, their areas are described by the same function $f(\theta,\phi)$. Note that $f(\theta,\phi)$ is universal, dimensionless, and scale-invariant.

Because the area of the original triangle A_c is equal to the sum of the combined A_a and A_b,

$$A_a + A_b = A_c, \tag{3}$$

we can substitute Eqs. (1) and (2) to obtain

$$a^2 f(\theta,\phi) + b^2 f(\theta,\phi) = c^2 f(\theta,\phi), \tag{4}$$

and hence

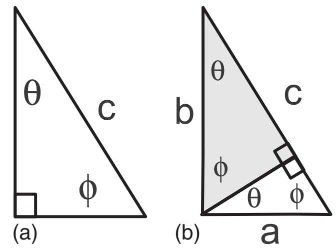

Fig. 1. (a) A right triangle specified by the angles $\{\theta,\phi\}$ and hypotenuse c. (b) The same triangular area can be described by two similar triangles of hypotenuses a and b.

$$a^2+b^2=c^2, \tag{5}$$

which is the Pythagorean theorem. There are much simpler methods to prove this theorem. However, this method does illustrate the power of dimensional analysis approach.[9] Additionally, we gain a new perspective on the Pythagorean theorem in this proof because it is linked to area calculations.

There are instances, such as renormalizable field theory, where dimensional analysis tools are essential to making certain calculations tractable. The following example will illustrate some of these features.

III. AN INFINITE LINE OF CHARGE

Consider the calculation of the electric potential V for the case of an infinite line of charge with constant linear charge density $\lambda=dQ/dy$. The contribution to the electric potential from an infinitesimal charge dQ is given by[14]

$$dV=\frac{1}{4\pi\epsilon_0}\frac{dQ}{r}. \tag{6}$$

We choose the coordinate system (see Fig. 2) so that x specifies the perpendicular distance from the wire, y is the coordinate along the wire, and $r=\sqrt{x^2+y^2}$. Given $\lambda=dQ/dy$ we have $dQ=\lambda dy$, and we can integrate along the length of the wire to obtain

$$V(x)=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{dy}{\sqrt{x^2+y^2}}=\infty. \tag{7}$$

This integral is logarithmically divergent and we obtain an infinite result.

If we take a closer look at this integral, we will see that it is scale-invariant. That is, if we rescale the argument x by a constant factor k $(x\to kx)$, the result is invariant,

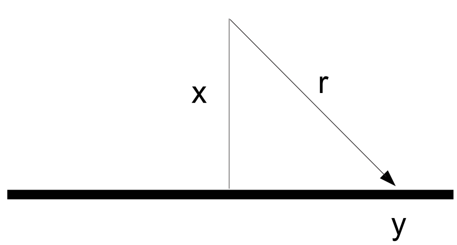

Fig. 2. Coordinate system for an infinite line of charge running in the y-direction with linear charge density $\lambda=dQ/dy$. We calculate the potential $V(x)$ at a fixed perpendicular distance x from the line of charge. The distance to the element of charge dQ is $r=\sqrt{x^2+y^2}$.

$$V(kx)=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dy\frac{1}{\sqrt{(kx)^2+y^2}}, \tag{8a}$$

$$=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}d(y/k)\frac{1}{\sqrt{x^2+(y/k)^2}}, \tag{8b}$$

$$=\frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}dz\frac{1}{\sqrt{x^2+z^2}}=V(x). \tag{8c}$$

In Eq. (8) we implemented the rescaling $z=y/k$. Because both y and z are dummy variables and the integration limits are infinite, the integral is unchanged. A consequence of this scale invariance is that

$$V(x_1)=V(x_2) \tag{9}$$

for any x_1 and x_2.

At first glance, this result appears to be a disaster because the usual purpose of the electric potential is to calculate the work W via the relation

$$W/Q=\Delta V=V(x_2)-V(x_1), \tag{10}$$

or to calculate the electric field using

$$\vec{E}=-\vec{\nabla}V. \tag{11}$$

Because $V(x_2)-V(x_1)=0$ [see Eq. (9)] our attempts to calculate the work W or the electric field $\vec{E}$ are meaningless.

We now understand why it is fortunate that $V(x)$ is infinite because infinite numbers have some unusual properties. For example, given a finite constant c we can write (schematically) $\infty+c=\infty$ which implies that $\infty-\infty=c$. Thus, even though $V(x_1)=V(x_2)$, we can still find that the difference is nonzero because these quantities are infinite: $\delta V=V(x_2)-V(x_1)\neq 0$. The challenge is that the difference of two infinite quantities is ambiguous. That is, how can we tell if $\infty-\infty=c_1$ or $\infty-\infty=c_2$ is the correct physical result? The solution is that we must regularize the infinite quantities so that we can uniquely extract the difference.

IV. CUTOFF REGULARIZATION

We first regularize the integral using a simple cutoff method. That is, instead of considering an infinite wire, we will calculate the potential for a finite wire of length $2L$. In this case the potential becomes[15]

$$V(x)=\frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{+L}dy\frac{1}{\sqrt{x^2+y^2}}, \tag{12a}$$

$$=\frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{+L+\sqrt{L^2+x^2}}{-L+\sqrt{L^2+x^2}}\right). \tag{12b}$$

We observe that the result is finite and that in addition to the physical length scale x, $V(x)$ depends on the artificial regulator L. We cannot remove the regulator L without $V(x)$ becoming singular, and the result for $V(x)$ violates a symmetry of the original problem—translation invariance.

Even though $V(x)$ depends on the artificial regulator L, we observe that all physical quantities are independent of this regulator in the limit $L\to\infty$. Specifically, for the electric field we have

Table I. Angular integration measure Ω_n as a function of the dimension n. The surface of the n-dimensional volume V_n is an $(n-1)$-dimensional manifold S_{n-1}. We recognize Ω_2 as the circumference of the unit circle, Ω_3 as the surface area of the unit sphere, and Ω_4 as the three-surface of the four-dimensional unit hypersphere. See the Appendix for details.

n	Ω_n	$\Gamma(n/2)$	Object	V_n	Surface	S_{n-1}
1	2	$\sqrt{\pi}$	Line	$2R$	Point	2
2	2π	1	Disk	πR^2	Line	$2\pi R$
3	4π	$\frac{1}{2}\sqrt{\pi}$	Three-ball	$\frac{4\pi}{3}R^3$	Two-sphere	$4\pi R^2$
4	$2\pi^2$	1	Four-ball	$\frac{\pi^2}{2}R^4$	Three-sphere	$2\pi^2 R^3$
5	$\frac{8\pi^2}{3}$	$\frac{3}{4}\sqrt{\pi}$	Five-ball	$\frac{8\pi^2}{15}R^5$	Four-sphere	$\frac{8\pi^2}{3}R^4$

$$E(x) = -\frac{\partial V(x)}{\partial x} = \frac{\lambda}{2\pi\epsilon_0 x}\frac{L}{\sqrt{L^2+x^2}}, \tag{13a}$$

$$\xrightarrow[L\to\infty]{} \frac{\lambda}{2\pi\epsilon_0 x}. \tag{13b}$$

For the potential difference (proportional to the electric work W) we have

$$\delta V = V(x_1) - V(x_2) \xrightarrow[L\to\infty]{} \frac{\lambda}{4\pi\epsilon_0}\ln\frac{x_2^2}{x_1^2}. \tag{14}$$

As we observed in Sec. II, δV is finite even though it is the difference of two infinite terms $V(x_1)$ and $V(x_2)$. The regulator L unambiguously allows us to extract the finite difference δV, at which point the regulator can be discarded. The fact that the physical quantities $E(x)$ and δV are independent of the unphysical regulator is an essential property of the regularization method. We will discuss this property further in Sec. VII.

The presence of the cutoff L breaks the translation symmetry of the original example. That is, for a truly infinite wire, our position in the y-direction is inconsequential. However, for a finite wire this is no longer the case. Specifically, if we shift our y-position by a constant c to $y \to y' = y + c$, we have

$$V(x) = \frac{\lambda}{4\pi\epsilon_0}\int_{-L+c}^{+L+c} dy \frac{1}{\sqrt{x^2+y^2}}, \tag{15a}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{+(L+c)+\sqrt{(L+c)^2+x^2}}{-(L-c)+\sqrt{(L-c)^2+x^2}}\right). \tag{15b}$$

We have lost the translation invariance $y \to y' = y + c$.

Although preserving symmetries is not of paramount importance in this simple example, it is essential for certain field theory calculations. We now repeat this calculation, but instead use dimensional regularization, which will preserve the translational symmetry.

V. DIMENSIONAL REGULARIZATION

The central idea of dimensional regularization is to calculate $V(x)$ in n-dimensions where n is not necessarily an integer.[3–5] We can generalize the integration of Eq. (7) by replacing the one-dimensional integration $\int dy$ by the general n-dimension result. Specifically, we make the replacement[16]

$$\int_{-\infty}^{+\infty} dy = \int dV_1 \to \int dV_n = \int d\Omega_n \int_0^{+\infty} y^{n-1} dy, \tag{16}$$

where the angular integration measure is given by

$$\Omega_n = \int d\Omega_n = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \equiv \frac{n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}. \tag{17}$$

Here, Ω_n is the solid-angle in n-dimensions, and we have used $\Gamma(z+1) = z\Gamma(z)$ where Γ is the Gamma function. In the Appendix we provide an additional explanation, and verify that Ω_n yields the expected results for integer dimensions as tabulated in Table I.

The generalized formula for $V(x)$ now reads[8]

$$V(x) = \frac{\lambda}{4\pi\epsilon_0}\int d\Omega_n \int_0^{+\infty} \frac{y^{n-1}}{\mu^{n-1}}\frac{dy}{\sqrt{x^2+y^2}}. \tag{18}$$

Note that we are forced to introduce an auxiliary scale factor of μ^{n-1}, where μ has units of length, to ensure $V(x)$ has the correct dimension.[17] We replace $n = 1 - 2\epsilon$ to facilitate expanding about $n = 1$ and obtain

$$V(x) = \frac{\lambda}{4\pi\epsilon_0}\frac{\Gamma\left(\frac{1-n}{2}\right)}{\left(\frac{x}{\mu}\sqrt{\pi}\right)^{1-n}}, \tag{19a}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{\mu^{2\epsilon}}{x^{2\epsilon}}\frac{\Gamma(\epsilon)}{\pi^{\epsilon}}\right). \tag{19b}$$

We make the following observations about the dimensionally regularized result: $V(x)$ depends on an artificial regulator ϵ which is dimensionless, and also depends on an auxiliary scale μ which has dimensions of length. If we remove either the regulator ϵ or the auxiliary scale μ, then $V(x)$ becomes ill-defined. The dimensional regularization preserves the translation invariance of the original problem.

It is interesting to contrast this result with the cutoff regularization method where L serves as both the regulator and the auxiliary scale.

For the potential difference we find

$$\delta V = V(x_1) - V(x_2) \xrightarrow[\epsilon\to 0]{} \frac{\lambda}{4\pi\epsilon_0}\ln\left(\frac{x_2^2}{x_1^2}\right), \tag{20}$$

and for the electric field we obtain

$$E = \frac{-\partial V(x)}{\partial x} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{2\epsilon\mu^{2\epsilon}\Gamma(\epsilon)}{\pi^{\epsilon}x^{1+2\epsilon}}\right], \tag{21a}$$

$$\xrightarrow[\epsilon\to 0]{} \frac{\lambda}{2\pi\epsilon_0}\frac{1}{x}. \tag{21b}$$

As before, we observe that all physical quantities are independent of both the regulator ϵ and the auxiliary scale μ.

In conclusion, we find that the problem for $V(x)$ is solved at the expense of an artificial regulator ϵ and an auxiliary scale μ. We also note that the regulator ϵ and auxiliary scale μ are separate entities, in contrast to the cutoff regularization method where the length L plays both roles. Additionally, translational invariance symmetry is preserved. The fact that dimensional regularization respects symmetries makes this technique indispensable for field theory calculations involving gauge symmetries and Lorentz symmetries.

VI. RENORMALIZATION

Having demonstrated two separate methods to regularize the infinities that enter the calculation of $V(x)$, we now turn to renormalization.

Physical quantities such as the work $W\approx\delta V$ and the electric field $\vec{E}=-\vec{\nabla}V$ are derived from $V(x)$, but the potential itself is not a physical quantity. In particular, we can shift the potential by a constant c, $V\to V+c$, and the physical quantities will be unchanged.

To illustrate this point, we expand $V(x)$ of Eq. (19b) in powers of ϵ,

$$V(x) = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} + \ln\left(\frac{e^{-\gamma_E}}{\pi}\right) + \ln\left(\frac{\mu^2}{x^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{22}$$

Here, $\gamma_E \simeq 0.577\,216$ is the Euler constant which arises from expanding the Gamma function $\Gamma(\epsilon)\sim -\gamma_E + 1/\epsilon$.

We now apply a minimal subtraction (MS) prescription. We have the freedom to shift $V(x)$ by a constant, and we choose this constant to eliminate the $1/\epsilon$ term,

$$V_{\mathrm{MS}}(x) = \frac{\lambda}{4\pi\epsilon_0}\left[+\ln\left(\frac{e^{-\gamma_E}}{\pi}\right) + \ln\left(\frac{\mu^2}{x^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{23}$$

We can go even further and invent a modified minimal subtraction ($\overline{\mathrm{MS}}$) prescription[18] to eliminate the $\ln(e^{-\gamma_E}/\pi)$ term as well,

$$V_{\overline{\mathrm{MS}}}(x) = \frac{\lambda}{4\pi\epsilon_0}\left[+\ln\left(\frac{\mu^2}{x^2}\right) + \mathcal{O}(\epsilon)\right]. \tag{24}$$

After renormalization we can remove the regulator ($\epsilon\to 0$), but not the auxiliary scale μ. Recall that without an auxiliary scale to generate a dimensionless ratio μ/x, we could not have any substantive x-dependence.

In addition to the μ-dependence, $V(x)$ depends on the renormalization scheme. However, physical observables must be independent of the auxiliary scale μ and the particular renormalization scheme. For example, the calculated potential differences yield identical results when calculated consistently in a single renormalization scheme,

$$V_{\mathrm{MS}}(x_1) - V_{\mathrm{MS}}(x_2) = \delta V = V_{\overline{\mathrm{MS}}}(x_1) - V_{\overline{\mathrm{MS}}}(x_2). \tag{25}$$

Here, the results of the MS and the modified minimal subtraction ($\overline{\mathrm{MS}}$) are identical for physical quantities.

However, if we mix renormalization schemes inconsistently, we will obtain nonsensical results that are dependent on the choice of scheme,[19]

$$V_{\overline{\mathrm{MS}}}(x_1) - V_{\mathrm{MS}}(x_2) \neq \delta V \neq V_{\mathrm{MS}}(x_1) - V_{\overline{\mathrm{MS}}}(x_2). \tag{26}$$

The elementary problem of the infinite line charge contains all the key concepts of dimensional regularization and renormalization that we encounter in quantum field theory radiative calculations. For example, in the radiative quantum chromodynamics (QCD) calculation of the Drell–Yan process $(q\bar{q}\to\gamma^*\to\mu^+\mu^-)$ we encounter the infinite expression[20,21]

$$\frac{D(\epsilon)}{\epsilon} = \left(\frac{4\pi\mu^2}{Q^2}\right)^{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}, \tag{27a}$$

$$\sim \frac{1}{\epsilon} - \ln\left(\frac{e^{+\gamma_E}}{4\pi}\right) + \ln\left(\frac{\mu^2}{Q^2}\right). \tag{27b}$$

In Eq. (27), Q represents the characteristic energy scale and is the independent variable that is analogous to x in our example. Although this expression is for a four-dimensional QCD calculation, the structure of the divergent term is remarkably similar to our simple one-dimensional example. For the QCD calculation, the minimal subtraction prescription for this Drell–Yan calculation eliminates the $1/\epsilon$ term, and the modified minimal subtraction prescription eliminates the $1/\epsilon - \ln[e^{+\gamma_e}/(4\pi)]$ term so that only the term $\ln[\mu^2/Q^2]$ remains.

VII. THE RENORMALIZATION GROUP EQUATION

The fact that the physical observables are independent of the unphysical auxiliary scale μ is a consequence of the renormalization group equation[22,23]

$$\mu\frac{d\sigma}{d\mu} = 0, \tag{28}$$

where σ represents any physical observable. Thus, the renormalization group equation implies that the electric field $\vec{E}=\vec{\nabla}V$ and the work $W=\delta V$ are also independent of the μ scale,

$$\mu\frac{dE}{d\mu} = 0, \qquad \mu\frac{dW}{d\mu} = 0. \tag{29}$$

These results are implicit in the final expression for the physical quantities E and V.

Although the result of Eq. (28) appears to be almost trivial, the renormalization group equation yields a very important result when applied to scattering processes involving nonperturbative hadronic particles (proton, nucleons, etc.). We can write the physical cross section σ as a product of a nonperturbative distribution f which describes the soft (low energy) physics, and a perturbative term ω which describes the hard (high energy) physics,[24]

$$\sigma = f\omega. \tag{30}$$

We differentiate with respect to $\ln \mu$ and apply the chain rule and find

$$\frac{d\sigma}{d \ln \mu} = 0 = \frac{df}{d \ln \mu}\omega + f\frac{d\omega}{d \ln \mu}, \tag{31}$$

where we have used Eq. (28). We rearrange terms and place all the f dependence on the left-hand-side and the ω dependence on the right-hand-side,

$$\frac{1}{f}\frac{df}{d \ln \mu} = -\gamma = \frac{1}{\omega}\frac{d\omega}{d \ln \mu}. \tag{32}$$

We next introduce the separation constant $-\gamma$.[25] The left-hand side of Eq. (32) depends only on the nonperturbative quantity f, and therefore, the left-hand side is (in principle) incalculable. Conversely, the right-hand side of Eq. (32) depends only on the perturbative quantity ω. Therefore, the right-hand side is calculable in perturbation theory. We can use this property to calculate $-\gamma$.

Having calculated $-\gamma$, we can solve Eq. (32) for f to obtain[26]

$$f \sim \mu^{-\gamma}. \tag{33}$$

Equation (33) is a remarkable result. Even though f is an incalculable nonperturbative quantity, we are able to find the μ-dependence of this function. That is, the renormalization group equation has allowed us to calculate the μ-dependence of an incalculable quantity by relating the (incalculable) nonperturbative df/f to the (calculable) perturbative $d\omega/\omega = -\gamma$.

VIII. EXTRA DIMENSIONS

In our example we used the trick of generalizing the number of integration dimensions from an integer to a continuous parameter. Although we only let the dimension stray by 2ϵ, it is useful to consider bigger shifts as in the case of extra dimensions which have recently been hypothesized.[27,28] In this section, we provide an example of a dimensional transmutation where the effective dimension D_{eff} changes from one integer to another as we probe the system at different scales.

For example, we can generalize the r-dependence of the potential and electric field for the case of D-dimensions as[29]

$$V(r) \sim \frac{1}{r^{D-2}}, \quad E(r) \sim \frac{1}{r^{D-1}}. \tag{34}$$

A quick check will verify that Eq. (34) reproduces the usual expressions in $D=3$ spatial dimensions. Additionally, in three dimensions we can create charge distributions that mimic lower order spatial dimensions (see Table II). For a (zero-dimensional) point-charge in three dimensions, the electric field lines spread out on a surface of $D-1=2$ dimensions according to Gauss's law, and we observe $E(r) \sim 1/r^2$. Similarly, for a (one-dimensional) line charge, space is now effectively $D=2$ dimensional, and hence the electric field lines spread out on a surface of $D-1=1$ dimension, and we observe $E(r) \sim 1/r$. Finally, for a (two-dimensional) sheet-charge, space is now effectively $D=1$ dimensional, and hence the electric field lines spread out on in $D-1=0$ dimensions, and we observe $E(r) \sim 1/r^0 = \text{const}$.

Table II. Example charge configurations that illustrate $D_{\text{eff}} = \{3, 2, 1\}$ effective dimensions.

D_{eff}	$E(r)$	$V(r)$	Example
3	$\frac{1}{r^2}$	$\frac{1}{r}$	Point charge
2	$\frac{1}{r^1}$	$\ln r$	Line charge
1	$\frac{1}{r^0}$	r	Sheet charge

Figure 3 displays the electric field lines for a point charge confined to one infinite dimension (x) and one finite (or compact) dimension (y) of scale R. We observe that if we examine the electric field at scales small compared to the compact dimension R ($r \ll R$), we find that the electric field lines spread out in two dimensions, and we obtain the usual two-dimensional result $E(r) \sim 1/r$. Conversely, if we examine the electric field at distance scales large compared to the compact dimension R ($r \gg R$), we find the one-dimensional result $E(r) \sim \text{const}$. In this example, the effective dimension of space changes as we move from small ($D=2$) to large length scales ($D=1$).

IX. CONCLUSIONS

We have calculated the potential of an infinite line of charge using dimensional regularization. By contrasting this calculation with the conventional cutoff approach, we demonstrated that dimensional regularization respects the symmetries of the problem, namely, translational invariance. Dimensional regularization requires that we introduce a regulator ϵ and an auxiliary length scale μ. We then renormalized the potential to eliminate the $1/\epsilon$ singularities. This potential was finite and independent of the regulator ϵ, but it depended on the particular renormalization scheme and renormalization scale μ. However, we demonstrated that all physical observables ($E, \delta V$) were scheme and scale-invariant.

Because this example exhibits many of the key features of dimensional regularization as applied to quantum field theory, it provides an excellent opportunity to understand the virtues of this regularization method without the complications of gauge symmetries. As such, this example serves as an ideal pedagogical study.

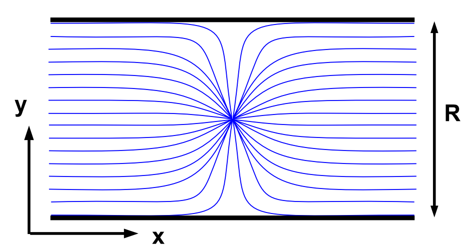

Fig. 3. Electric field of a point charge confined in one infinite dimension (x) and one finite dimension (y) of scale R.

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APPENDIX: VOLUME AND AREA IN ARBITRARY DIMENSIONS

The volume V_3 of a three-sphere in spherical coordinates is a product of the angular and radial integrals,

$$V_3 = \int d\Omega_3 \int_0^R r^2 dr = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta d\theta \int_0^R r^2 dr = \frac{4\pi}{3} R^3. \tag{A1}$$

The angular integral $\int \Omega_3$ is dimensionless, and the radial integral $\int r^2 dr$ carries the dimensions.

For the two-dimensional surface area S_2 we can use the V_3 integral with a δ-function $\delta(r-R)$ to constrain us to the surface

$$S_2 = \int d\Omega_3 \int_0^R dr r^2 \delta(r-R) = 4\pi R^2. \tag{A2}$$

Having established the familiar three-dimensional case, we can generalize to n-dimensions,

$$V_n = \int d\Omega_n \int_0^R r^{n-1} dr = \Omega_n \frac{R^n}{n}, \tag{A3}$$

and the $(n-1)$-dimensional surface area (S_{n-1}) of the n-dimensional volume V_n is

$$S_{n-1} = \int d\Omega_n \int_0^R dr r^{n-1} \delta(r-R) = \Omega_n R^{n-1}. \tag{A4}$$

With Eqs. (A3) and (A4) we have the general relation

$$\frac{V_n}{S_{n-1}} = \frac{R}{n}. \tag{A5}$$

Additionally, we find

$$\frac{dV_n}{dR} = S_{n-1}. \tag{A6}$$

Equation (A6) demonstrates that the derivative (or boundary) of the volume is the surface area.

Because the one-dimensional case has a subtle factor of 2, we calculate this case explicitly. We use Eq. (A3) to find the volume of a line to be

$$V_1 = \int dV_1 = \int d\Omega_1 \int_0^R r^0 dr = 2R. \tag{A7}$$

This result is not R but $2R$ because the one-dimensional line extends from $-R$ to $+R$.

In the notation of Eq. (7) we have (with $R \to \infty$)

$$\int dV_1 = \int d\Omega_1 \int_0^{+\infty} dy = 2\int_0^{+\infty} dy = \int_{-\infty}^{+\infty} dy. \tag{A8}$$

We can make the replacement $\int_{-\infty}^{+\infty} dy \to \int dV_1$, and the n-dimensional generalization is

$$\int_{-\infty}^{+\infty} dy = \int dV_1 \to \int dV_n = \int d\Omega_n \int_0^{+\infty} y^{n-1} dy. \tag{A9}$$

Equation (7) for the potential $V(x)$ becomes

$$V(x) = \frac{\lambda}{4\pi\epsilon_0} \int d\Omega_n \int_0^{+\infty} y^{n-1} \frac{dy}{\sqrt{x^2+y^2}}. \tag{A10}$$

Note that Eq. (A10) is not dimensionally correct because the factor y^{n-1} needs to be compensated by introducing an auxiliary scale factor as we did in Eq. (18).

[1] Gloria B. Lubkin, "Nobel Prize to 't Hooft and Veltman for putting electroweak theory on firmer foundation," Phys. Today **52** (12), 17–19 (1999).

[2] See Ref. 1 and the webpage citation for the 1999 Nobel Prize in Physics at ⟨nobelprize.org/⟩.

[3] Gerard 't Hooft and M. J. G. Veltman, "Regularization and renormalization of gauge fields," Nucl. Phys. B **44**, 189–213 (1972).

[4] Gerard 't Hooft, "Dimensional regularization and the renormalization group," Nucl. Phys. B **61**, 455–468 (1973).

[5] C. G. Bollini and J. J. Giambiagi, "Dimensional renormalization: The number of dimensions as a regularizing parameter," Nuovo Cimento B **12**, 20–25 (1972).

[6] For chiral symmetries there are some subtle difficulties that must be handled carefully. In particular, the properties of the parity operator depend on the dimensionality of space-time.

[7] C. Kaufman, "An illustration from classical physics of renormalization mathematics," Am. J. Phys. **37**, 560–561 (1969).

[8] M. Hans, "An electrostatic example to illustrate dimensional regularization and renormalization group technique," Am. J. Phys. **51**, 694–698 (1983).

[9] In Sec. V we will use dimensional analysis to demonstrate that we must introduce an auxiliary scale μ in addition to the regulator ϵ. For other interesting applications of scaling and dimensional analysis, see Refs. 10–13.

[10] Arkady B. Migdal, *Qualitative Methods in Quantum Theory* (Westview Press, Boulder, CO, 2000), pp. 1–437.

[11] Steven Vogel, "Exposing life's limits with dimensionless numbers," Phys. Today **51** (11), 22–27 (1998).

[12] Steven Vogel, *Cats' Paws and Catapults: Mechanical Worlds of Nature and People* (Norton, New York, 2000), pp. 1–384.

[13] Matt A. Bernstein and William A. Friedman, *Thinking About Equations: A Practical Guide for Developing Mathematical Intuition in the Physical Sciences and Engineering* (John Wiley & Sons, New York, 2009), pp. 1–258.

[14] We will use SI units so that our results reduce to the usual undergraduate textbook expressions.

[15] For simplicity, we will calculate the potential at the midpoint of the wire. The general case is more complicated algebraically, but yields the same result in the $L \to \infty$ limit.

[16] Here, V_n with a subscript represents volume, and $V(x)$ represents the potential.

[17] Because the factor $\lambda/(4\pi\epsilon_0)$ has units of potential, the integral must be dimensionless.

[18] In the literature, this modified minimal subtraction scheme is commonly identified as the $\overline{\text{MS}}$ (MS-bar) scheme.

[19] The reader is invited to verify that the calculation of the electric field $\vec{E}(x)$ in a consistent renormalization scheme yields the results of Eq. (21b), and an inconsistent application of the schemes does not.

[20] B. Potter, "Calculational techniques in perturbative QCD: The Drell-Yan process," ⟨citeseer.ist.psu.edu/209991.html⟩.

[21] See Ref. 20, Eqs. (46) and (47).

[22] Bertrand Delamotte, "A hint of renormalization," Am. J. Phys. **72**, 170–184 (2004).
[23] For an excellent pedagogical analysis of the renormalization group equation, see Ref. 22.
[24] More precisely, f is a "parton distribution function," and ω is a "hard-scattering cross section." The cross section σ is a convolution $\sigma = f \otimes \omega$ which can be decomposed by taking Mellin moments. Hence, the discussion of this section applies formally to the Mellin transforms of f and ω.
[25] Unless f and ω are trivially related, the most reasonable solution for this type of differential equation is that both the left-hand and right-hand sides of Eq. (32) equal a separation constant, $-\gamma$.
[26] The term $-\gamma$ is referred to as the *anomalous dimension*. It is a dimension because it determines the μ-scaling dimension of f in Eq. (33). It is anomalous because if f satisfied exact scaling, f would be invariant under a scale change $\mu_1 \to \mu_2$, so that $f = \mu^0 = \text{const}$, and any nonzero value for $-\gamma$ would be anomalous.
[27] Nima Arkani-Hamed, Savas Dimopoulos, and G. R. Dvali, "The hierarchy problem and new dimensions at a millimeter," Phys. Lett. B **429**, 263–272 (1998).
[28] Lisa Randall and Raman Sundrum, "A large mass hierarchy from a small extra dimension," Phys. Rev. Lett. **83**, 3370–3373 (1999).
[29] Note that for the special case $D=2$ the potential $V(r)$ has a logarithmic form. See Table II for details.

Ford Coil. When the PSSC apparatus was first produced in the late 1950s, the high voltage needed to actuate discharge coils was produced by apparatus from the past a Ford Coil. These were used in the ignition systems of Model T Ford automobiles that were produced from 1914 to 1927. This coil is marked Ford in the typical flowing script, and turns 6 V DC into about 10 KV AC. The mechanism is just the same as the 19th century induction coil, with an electro-magnetic make-and-break contact to turn DC into pulsating DC in the primary, and an internal capacitor to eliminate sparking at the contacts. It was made ca. 1914 and is in the Greenslade collection. (Notes and photograph by Thomas B. Greenslade, Jr., Kenyon College.)