Energy-based stability criterion for the Finite-Difference Time-Domain method

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Abstract

This paper presents an energy-based stability analysis for the classical FDTD scheme using Lyapunov functions. Following a short introduction on the Lyapunov functions, the application of the discrete counterpart is used to derive the well-known stability criterion for the FDTD method. Lossy or anisotropic (i.e. biaxial symmetry) media are seamless integrable into the derivation.

Introduction

Upcoming problem settings in the realm of e.g. microwave engineering are constantly growing in complexity, size and needs. Especially the complexity of dispersive materials and the usage of natural or artificial models into layered structures make it crucial for upcoming numerical methods to have the appropriate stability criterion. A common and versatile time domain method computational electromagnetics is the FDTD method. The first stability criterion was derived by Taflove [1]. An extension to higher order FDTD schemes were computed by Fang [2], whereas Liu [3] broaden the view to different collocation schemes. All mentioned authors used the von Neumann method, which suffers from difficulties when embedding dispersive media. A way out arise through the involvement of the Routh-Hurwitz criterion by Pereda [4]. Both methods include an inevitable transformation process where it becomes difficult to apply to subgridding or hybrid schemes. A new approach was proposed by Kung [5], who incorporates discretized Lyapunov functions, but the derived criterion limits the time-step stronger than necessary. An extension which leads to the known Courant limit, were made by Edelvik [6], but his approach could become misleading for higher order FDTD schemes. In this paper we give a short introduction into the energy-based stability criterion derivation, which is basically capable to compute the criterions for all higher order schemes. Due to the limited space the derivation of the most common second order approximation is presented here.

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Lyapunov functions

The stability of dynamical and discrete systems such as the FDTD scheme can be computed with the help of the Lyapunov matrix $\boldsymbol{A}$.

$$V_{cont.}(x) \qquad \Rightarrow \qquad V_{disc.}(x) = \vec{x}^T \mathbf{A}\, \vec{x}$$

$$\vec{x} = (x_0, x_1, \cdots, x_n)^T \qquad \mathbf{A} = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix} \tag{1}$$

The dynamic system is stable, if the square matrix $\boldsymbol{A}$ is positive definite, which is valid if all eigenvalues or all determinants associated with all upper-left sub-matrices are positive. This is known as the Sylvester's criterion [7].

Poyntings Theorem

The theorem displays the conservation of energy for electromagnetic fields. The total energy W_{total} could be defined as a Lyapunov function and the discretized counterpart should therefore be proven as a positive definite matrix. Starting with Ampere's law, we yield

$$\begin{aligned} \vec{J}^{n+.5} + \partial_t \vec{D}^{n+.5} &= \nabla \times \vec{H}^{n+.5} \\ \vec{E}^{n+.5} \cdot \left(\vec{J}^{n+.5} + \partial_t \vec{D}^{n+.5}\right) &= 0.5 \cdot (E^{n+1} + E^n) \cdot \nabla \times \ \vec{H}^{n+.5} \\ \vec{E}^{n+.5} \vec{J}^{n+.5} + \frac{\epsilon}{2\Delta t}(|\vec{E}^{n+1}|^2 - |\vec{E}^n|^2) &= \vec{E}^{n+.5} \cdot \nabla \times \ \vec{H}^{n+.5} \end{aligned} \tag{2}$$

The Faraday's law can be derived in a similar manner and the subtraction of both equations yields after some simplifications into

$$\begin{aligned} &2 \iiint\limits_V \left(\vec{E}^{n+.5} \vec{J}^{n+.5} + \vec{H}^n \vec{M}^n + \nabla \cdot \left(\vec{E}^{n+.5} \times \vec{H}^{n+.5}\right)\right) dV = \\ &= -\frac{1}{\Delta t} \iiint\limits_V \Bigg[\left(\epsilon \left|\vec{E}^{n+1}\right|^2 + \mu \left|\vec{H}^{n+.5}\right|^2 - \Delta t\, \vec{H}^{n+.5} \cdot \nabla \times \ \vec{E}^{n+1}\right) - \\ &\quad - \left(\epsilon \left|\vec{E}^{n}\right|^2 + \mu \left|\vec{H}^{n-.5}\right|^2 - \Delta t\, \vec{H}^{n-.5} \cdot \nabla \times \ \vec{E}^{n}\right)\Bigg] dV \\ &= -\frac{1}{\Delta t} \iiint\limits_V \left[(w_x + w_y + w_z)^{(n+.5)\Delta t \le t \le (n+1)\Delta t} - w^{(n-.5)\Delta t \le t \le n\Delta t}\right] dV \end{aligned} \tag{3}$$

Equation (3) describes the decay of energy through power dissipation for the discrete system across the entire simulation volume. The discretization of the volume integral carries the capability to rearrange all field components inside the summation to the Yee collocation scheme without violating the energy conservation. The cause of the energy flow is described by the underlying dynamics of the Maxwell equations, whereas the action is depicted by equation (3) and describes the Lyapunov function space. It is important to mention the additional curl operation in equation (3), which occurs during discretization, vanishes in the case of infinite small time-steps and translates under that circumstance to the well-known continuous Poyntings theorem. Since an instable system tends towards infinite energy, the discretized theorem provides an ideal observation approach. The aforementioned rearrangement of the summands in equation (4) follows the access to the corresponding field components of the discretized Faradays law. Starting with the magnetic field component H_y at the index *(i,j,k),* all other non-shifted indexes are suppressed.

$$H_y^{n+.5} = H_y^{n-.5} - \frac{\Delta t}{\mu_y}\left(\frac{E_{x(k+1)} - E_x}{\Delta z} - \frac{E_{z(i+1)} - E_z}{\Delta x}\right) \tag{4}$$

The FDTD update scheme indicates that the magnetic field component H_y takes the entire energy from the electrical curl operation and the magnetic energy of the prior time-step. The appropriate energy-density w_y is centralized by the necessary rearrangement of the summands in equation (4) and follows

$$\begin{aligned}
w_y &= \mu_y \left|H_y^{n+.5}\right|^2 + \epsilon_{x(k+1)} \left|E_{x(k+1)}^{n+1}\right|^2 + \epsilon_x \left|E_x^{n+1}\right|^2 + \epsilon_{z(i+1)} \left|E_{z(i+1)}^{n+1}\right|^2 + \\
&\quad + \epsilon_z \left|E_z^{n+1}\right|^2 - H_y^{n+.5}\Delta t \left(\frac{E_{x(k+1)} - E_x}{\Delta z} - \frac{E_{z(i+1)} - E_z}{\Delta x}\right) \\
&= \begin{pmatrix} H_y \\ E_{x(k+1)} \\ E_x \\ E_{z(i+1)} \\ E_z \end{pmatrix}^T \begin{pmatrix} \mu_y & -\frac{\Delta t}{2\Delta z} & +\frac{\Delta t}{2\Delta z} & +\frac{\Delta t}{2\Delta x} & -\frac{\Delta t}{2\Delta x} \\ -\frac{\Delta t}{2\Delta z} & \epsilon_{x(k+1)} & 0 & 0 & 0 \\ +\frac{\Delta t}{2\Delta z} & 0 & \epsilon_x & 0 & 0 \\ +\frac{\Delta t}{2\Delta x} & 0 & 0 & \epsilon_{z(i+1)} & 0 \\ -\frac{\Delta t}{2\Delta x} & 0 & 0 & 0 & \epsilon_z \end{pmatrix} \begin{pmatrix} H_y \\ E_{x(k+1)} \\ E_x \\ E_{z(i+1)} \\ E_z \end{pmatrix} \\
&= \vec{x}^T \mathbf{A}_{xz} \vec{x}
\end{aligned} \tag{5}$$

Equation (5) depicts one of three crucial subsystems. It should be clearly mentioned that the system matrix $\boldsymbol{A}_{xz}$ is different to the reported ones in [5, 6] and carries no special energy distribution. The whole Lyapunov function is summarized into a 15 x 15 square matrix $\boldsymbol{A}$, where $\boldsymbol{0}$ represents a 5 x 5 matrix with zero values.

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{zy} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_{xz} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{A}_{yx} \end{pmatrix} \tag{6}$$

The positive definite property of the matrix $\boldsymbol{A}$, namely the stability of the FDTD scheme is fulfilled if the remaining time-step Δt is properly chosen under the condition that all determinants associated with all upper-left sub-matrices are positive. In the case of anisotropic (i.e. biaxial symmetry) and isotropic material the desired time-step yields into

$$\begin{aligned}
0 < \Delta t &\leq \frac{\sqrt{2}}{\sqrt{\frac{1}{\epsilon_x}\left(\frac{1}{\mu_y \Delta z^2} + \frac{1}{\mu_z \Delta y^2}\right) + \frac{1}{\epsilon_y}\left(\frac{1}{\mu_x \Delta z^2} + \frac{1}{\mu_z \Delta x^2}\right) + \frac{1}{\epsilon_z}\left(\frac{1}{\mu_x \Delta y^2} + \frac{1}{\mu_y \Delta x^2}\right)}} \\
0 < \Delta t &\leq \frac{1}{c\sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}}
\end{aligned} \tag{7}$$

The stability was verified via energy observation by a randomly filled biaxial permittivity and permeability loaded resonator depicted in Fig. 1.

Conclusion

The presented stability criterion has proven to be suitable as an alternative derivation for the known Courant limit of the FDTD method. As already mentioned, the von Neumann method or the extension via Routh-Hurwitz criterion needs an inevitable transformation process into the fourier or laplace domain, whereas the energy-based stability criterion remains completely in time domain. The versatile approach could be extended to embed the approaches for active [5], lossy or lumped [6] elements seamlessly. The well-known Courant limits for higher order schemes are computable with proper coefficients, but could not be expressed here due to the lack of space. Mixed order schemes or even upcoming hybrid schemes could be feasible within the presented approach.

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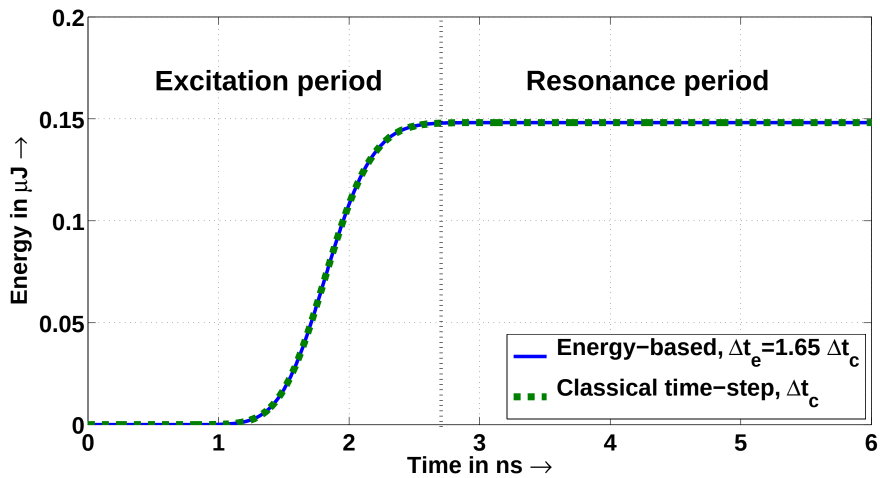

Figure 1: Energy over time inside a randomly filled biaxial permittivity and permeability loaded resonator in dependence of the chosen time-step.