Application of the Energy-based Stability Condition on the Spatial Wavelet-transformed FDTD Scheme

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Abstract: This paper presents an energy-based stability analysis on the classical FDTD scheme using so-called Lyapunov functions. Following a short introduction on a spatial wavelet-transformed FDTD method which has multi-resolution capabilities and notes that only neglecting wavelet coefficients in the specific resolution levels makes the scheme highly productive in comparison to the classical scheme. The fact of negligence makes the scheme inaccessible to classical stability analysis. The energy-based stability analysis is able to derive a condition and we prove their flexibility by application on the spatial wavelet-transformed FDTD scheme with and without neglected coefficients.

Keywords: WT-FDTD, spatial wavelet-transformed FDTD, MRSD, Multi-Resolution Spatial Domain, stability condition, poynting theorem, energy-based stability condition, Lyapunov functions.

1. Introduction

The FDTD method got the ability to solve many problems in the domain of computational electromagnetics with accurate results. It is extended over the years, but the progress of the method will always be limited by computer resources and the well known stability condition. The condition occurs during conversion from analytical equations into numerical procedures. To maintain stable and reliable solutions the numerical calculation procedure needs to be stable. There is an upper border for the time step Δt , starting from where the procedure becomes unstable and leads to non-physical results. In the literature one finds many approaches to derive the stability condition for different material properties [1, 2, 3, 4, 5, 6]. Kung [7, 8] published a new approach by using the poynting's theorem and derived a stability condition through conservation of energy for the computational simulation window. The derivation is made with the theory of stability using so-called Lyapunov functions. The usage of power flow or energy makes the formulation more flexible to different applications which were not accessible to some schemes. The resulting stability condition is much stricter than the classical one and has the possibility to include nonlinear behaviors. The nonlinear behavior plays an important role to a branch off the classical FDTD scheme firstly presented by Werthen and Wolff [9, 10]. Here the linear operator coefficients are set up in a FDTD manner and afterwards a discrete wavelet-transform is used to obtain the coefficients for the different scales or resolutions. The wavelet-transformed FDTD scheme inherits all advantages of the classical scheme and adds the capability of multi-resolution. Exclusively the negligence of the smallest wavelet-coefficients makes the method numerical efficient [11], but the resulting property inside the “leap-frog” calculation core is mainly nonlinear. In this paper we present an approach to derive stability conditions for wavelet-transformed FDTD schemes by using Kung’s formulation. Walter [12] used the norm of the linear operator and was nearly inaccessible to apply on very high complex operators which had many scales and transformation directions. The organization of this paper is divided into a short introduction of Lyapunov functions, the derivation of the stability condition for the classical FDTD scheme, the presentation and the solution of the stability condition on the wavelet-transformed FDTD. The last section gives a short summary.

2. Lyapunov functions

Alexander Lyapunov derived 1889 the mathematical formulation for stability analysis of ordinary differential equations. Their application is mainly found in the field of mechanical systems and had been applied by Kung [7,8] in the field of computational electromagnetics and adapted afterwards by A. Rennings to derive stability for meta-materials [13]. The rest position $\vec{x}_r$ is called Lyapunov stable, if small distortions of the strength δ around the position do not pass a specific value ε. The analysis of the stability begins by defining a dynamical system of the form $\dot{\vec{x}} = f(\vec{x})$ with the rest position $\vec{x}_r$. If the description $f(\vec{x}) = V(\vec{x})$ got the following properties

$$(1)\ V(\vec{x}_r) = 0, \qquad (2)\ V(\vec{x}) > 0 \wedge \vec{x} \neq \vec{x}_r, \qquad (3)\ \dot{V}(x) = \dot{\vec{x}}^T \nabla \cdot V(\vec{x}) < 0$$

The first and second equations define a minimum of the function at $\vec{x}_r$ and the third one asymptotically decays the Lyapunov function over the time. The function $V(\vec{x})$ can be expressed into a square matrix $\vec{A}$ by $V(\vec{x}) = \vec{x}^T \vec{A} \vec{x} > 0$, where the matrix $\vec{A}$ has to be positive definite to express a stable behavior. A positive definite matrix has only positive eigenvalues or instead of the eigenvalues all determinants associated with all upper-left sub-matrices of $\vec{A}$ are positive (Criterion of Sylvester) [14].

3. Poyntings theorem

The theorem displays the conservation of energy for electromagnetic fields.

$$\oiint (\vec{E} \times \vec{H}) d\vec{A} + P_L = -\frac{\partial}{\partial t}(W_{el} + W_{magn})$$

$$P_{total} = -\frac{\partial}{\partial t} W_{total} < 0 \quad \text{(after excitation)} \tag{4}$$

The total energy W_{total} could be attempted as a Lyapunov function and satisfy all named conditions if the energy W_{total} can be described as a positive definite matrix. The conversion of the analytical poynting theorem to a numerical form is straightforward, except of the discrete time-step. Starting with the Ampere’s law

$$\partial_t \vec{D}^{n+\frac{1}{2}} = \nabla \times \vec{H}^{n+\frac{1}{2}}$$

$$\vec{E}^{n+\frac{1}{2}} \cdot \partial_t \vec{D}^{n+\frac{1}{2}} = \vec{E}^{n+\frac{1}{2}} \cdot \nabla \times \vec{H}^{n+\frac{1}{2}}$$

$$\left(\frac{\vec{E}^{n+1} + \vec{E}^n}{2}\right) \cdot \left(\varepsilon \frac{\vec{E}^{n+1} - \vec{E}^n}{\Delta t}\right) = \left(\frac{\vec{E}^{n+1} + \vec{E}^n}{2}\right) \cdot \nabla \times \vec{H}^{n+\frac{1}{2}}$$

$$\frac{\varepsilon}{\Delta t}\left(\left|\vec{E}^{n+1}\right|^2 - \left|\vec{E}^n\right|^2\right) = \vec{E}^{n+1} \cdot \nabla \times \vec{H}^{n+\frac{1}{2}} + \vec{E}^n \cdot \nabla \times \vec{H}^{n+\frac{1}{2}} \tag{5}$$

and the Faraday's law of induction

$$-\partial_t \vec{B}^n = \nabla \times \vec{E}^n$$

$$-\vec{H}^n \cdot \partial_t \vec{B}^n = \vec{H}^n \cdot \nabla \times \vec{E}^n$$

$$-\left(\frac{\vec{H}^{n+\frac{1}{2}} + \vec{H}^{n-\frac{1}{2}}}{2}\right) \cdot \left(\mu \frac{\vec{H}^{n+\frac{1}{2}} - \vec{H}^{n-\frac{1}{2}}}{\Delta t}\right) = \left(\frac{\vec{H}^{n+\frac{1}{2}} + \vec{H}^{n-\frac{1}{2}}}{2}\right) \cdot \nabla \times \vec{E}^n$$

$$-\frac{\mu}{\Delta t}\left(\left|\vec{H}^{n+\frac{1}{2}}\right|^2 - \left|\vec{H}^{n-\frac{1}{2}}\right|^2\right) = \vec{H}^{n+\frac{1}{2}} \cdot \nabla \times \vec{E}^n + \vec{H}^{n-\frac{1}{2}} \cdot \nabla \times \vec{E}^n \tag{6}$$

Two complete sequences at $\vec{E}^{n+1} / \vec{H}^{n+\frac{1}{2}}$ and $\vec{E}^n / \vec{H}^{n-\frac{1}{2}}$ can be formulated with some mixed time function $\vec{E}^n \cdot \nabla \times \vec{H}^{n+\frac{1}{2}}$ and $\vec{H}^{n+\frac{1}{2}} \cdot \nabla \times \vec{E}^n$. We yield with using vector identity $\nabla \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot \nabla \times \vec{a} - \vec{a} \cdot \nabla \times \vec{b}$ and subtracting equation (6) from (5)

$$
\begin{aligned}
&\frac{\varepsilon}{\Delta t}\left(\left|\vec{E}^{n+1}\right|^2 - \left|\vec{E}^n\right|^2\right) + \frac{\mu}{\Delta t}\left(\left|\vec{H}^{n+\frac{1}{2}}\right|^2 - \left|\vec{H}^{n-\frac{1}{2}}\right|^2\right) = \\
&\qquad = \vec{E}^{n+1} \cdot \nabla \times \vec{H}^{n+\frac{1}{2}} + \vec{E}^n \cdot \nabla \times \vec{H}^{n+\frac{1}{2}} - \left(\vec{H}^{n+\frac{1}{2}} \cdot \nabla \times \vec{E}^n + \vec{H}^{n-\frac{1}{2}} \cdot \nabla \times \vec{E}^n\right) \\
&\qquad = -\nabla \cdot (\vec{E}^{n+1} \times \vec{H}^{n+\frac{1}{2}}) + \vec{H}^{n+\frac{1}{2}} \cdot \nabla \times \vec{E}^{n+1} - \nabla \cdot (\vec{E}^n \times \vec{H}^{n+\frac{1}{2}}) - \vec{H}^{n-\frac{1}{2}} \cdot \nabla \times \vec{E}^n \\
&\Leftrightarrow \nabla \cdot (\vec{E}^{n+1} \times \vec{H}^{n+\frac{1}{2}}) + \nabla \cdot (\vec{E}^n \times \vec{H}^{n+\frac{1}{2}}) = 2\nabla \cdot (\vec{E}^{n+\frac{1}{2}} \times \vec{H}^{n+\frac{1}{2}}) = 2\, p_{total}^{n+\frac{1}{2}} \\
&\qquad = -\frac{\varepsilon}{\Delta t}\left(\left|\vec{E}^{n+1}\right|^2 - \left|\vec{E}^n\right|^2\right) - \frac{\mu}{\Delta t}\left(\left|\vec{H}^{n+\frac{1}{2}}\right|^2 - \left|\vec{H}^{n-\frac{1}{2}}\right|^2\right) + \vec{H}^{n+\frac{1}{2}} \cdot \nabla \times \vec{E}^{n+1} - \vec{H}^{n-\frac{1}{2}} \cdot \nabla \times \vec{E}^n \\
&\qquad = -\frac{1}{\Delta t}\left[\left(\varepsilon\left|\vec{E}^{n+1}\right|^2 + \mu\left|\vec{H}^{n+\frac{1}{2}}\right|^2 - \Delta t \vec{H}^{n+\frac{1}{2}} \cdot \nabla \times \vec{E}^{n+1}\right) - \left(\varepsilon\left|\vec{E}^n\right|^2 + \mu\left|\vec{H}^{n-\frac{1}{2}}\right|^2 - \Delta t \vec{H}^{n-\frac{1}{2}} \cdot \nabla \times \vec{E}^n\right)\right] \\
&\qquad = -\left(\frac{w_{total}^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1} - w_{total}^{n \le \frac{t}{\Delta t} \le n+\frac{1}{2}}}{\Delta t}\right) \\
&\Leftrightarrow p_{total}^{n+\frac{1}{2}} = -\left(\frac{w_{total}^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1} - w_{total}^{n \le \frac{t}{\Delta t} \le n+\frac{1}{2}}}{2\Delta t}\right) \le 0 \quad \text{(after excitation)}
\end{aligned} \tag{7}
$$

The argument for positive definiteness of the energy density in one cell during the specific time-range $n \le \frac{t}{\Delta t} \le n + \frac{1}{2}$ is sufficient to imply the extension of this property to all cells for all time-steps.

$$
\begin{aligned}
W_{total}^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1} &= \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sum_{k=1}^{N_z} \Delta V_{(i,j,k)} \cdot w_{(i,j,k)}^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1} \\
w_{(i,j,k)}^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1} &= \left[\varepsilon\left(E_x^{n+1}\right)^2 + \mu\left(H_x^{n+\frac{1}{2}}\right)^2 - \Delta t \cdot H_x^{n+\frac{1}{2}}\left(\partial_y E_z^{n+1} - \partial_z E_y^{n+1}\right)\right] + \\
&\quad + \left[\varepsilon\left(E_y^{n+1}\right)^2 + \mu\left(H_y^{n+\frac{1}{2}}\right)^2 - \Delta t \cdot H_y^{n+\frac{1}{2}}\left(\partial_z E_x^{n+1} - \partial_x E_z^{n+1}\right)\right] + \\
&\quad + \left[\varepsilon\left(E_z^{n+1}\right)^2 + \mu\left(H_z^{n+\frac{1}{2}}\right)^2 - \Delta t \cdot H_z^{n+\frac{1}{2}}\left(\partial_x E_y^{n+1} - \partial_y E_x^{n+1}\right)\right] \\
&= w_x^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1} + w_y^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1} + w_z^{n+\frac{1}{2} \le \frac{t}{\Delta t} \le n+1}
\end{aligned} \tag{8}
$$

The preparation of the specific energy density into a symmetrical square matrix can be done by rearrange specific components without losing conservation of energy across the computational simulation window. Using the correlation of the discretized curl-operator to the neighbor cells allows this mathematical formulation. In the case of a second order derivative approximation the central difference operator use two components, for example $E^{n+1}_{z,(i,j+1,k)}, E^{n+1}_{z,(i,j,k)}$ and the access to these two components is twice by two derivative operators ∂_y, ∂_x which mean that one field-component is used four times in one energy cell. At the boundaries the access decreases from three to two times. The rearrangement of the energy-density on the condition of equal volume cells yields into

$$\begin{aligned}\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{total,(i,j,k)} &= \left[\begin{array}{l}\mu_x\left(H^{n+\frac{1}{2}}_{x,(i,j,k)}\right)^2+\frac{\varepsilon_z}{4}\left(E^{n+1}_{z,(i,j+1,k)}\right)^2+\frac{\varepsilon_z}{4}\left(E^{n+1}_{z,(i,j,k)}\right)^2+\frac{\varepsilon_y}{4}\left(E^{n+1}_{y,(i,j,k+1)}\right)^2+\frac{\varepsilon_y}{4}\left(E^{n+1}_{y,(i,j,k)}\right)^2- \\ -\Delta t\cdot H^{n+\frac{1}{2}}_{x,(i,j,k)}\left(\frac{E^{n+1}_{z,(i,j+1,k)}-E^{n+1}_{z,(i,j,k)}}{\Delta y}-\frac{E^{n+1}_{y,(i,j,k+1)}-E^{n+1}_{y,(i,j,k)}}{\Delta z}\right)\end{array}\right]+ \\ &+\left[\begin{array}{l}\mu\left(H^{n+\frac{1}{2}}_{y,(i,j,k)}\right)^2+\frac{\varepsilon_x}{4}\left(E^{n+1}_{x,(i,j,k+1)}\right)^2+\frac{\varepsilon_x}{4}\left(E^{n+1}_{x,(i,j,k)}\right)^2+\frac{\varepsilon_z}{4}\left(E^{n+1}_{z,(i+1,j,k)}\right)^2+\frac{\varepsilon_z}{4}\left(E^{n+1}_{z,(i,j,k)}\right)^2- \\ -\Delta t\cdot H^{n+\frac{1}{2}}_{y,(i,j,k)}\left(\frac{E^{n+1}_{x,(i,j,k+1)}-E^{n+1}_{x,(i,j,k)}}{\Delta z}-\frac{E^{n+1}_{z,(i+1,j,k)}-E^{n+1}_{z,(i,j,k)}}{\Delta x}\right)\end{array}\right]+ \\ &+\left[\begin{array}{l}\mu\left(H^{n+\frac{1}{2}}_{z,(i,j,k)}\right)^2+\frac{\varepsilon_y}{4}\left(E^{n+1}_{y,(i+1,j,k)}\right)^2+\frac{\varepsilon_y}{4}\left(E^{n+1}_{y,(i,j,k)}\right)^2+\frac{\varepsilon_x}{4}\left(E^{n+1}_{x,(i,j+1,k)}\right)^2+\frac{\varepsilon_x}{4}\left(E^{n+1}_{x,(i,j,k)}\right)^2- \\ -\Delta t\cdot H^{n+\frac{1}{2}}_{z,(i,j,k)}\left(\frac{E^{n+1}_{y,(i+1,j,k)}-E^{n+1}_{y,(i,j,k)}}{\Delta x}-\frac{E^{n+1}_{x,(i,j+1,k)}-E^{n+1}_{x,(i,j,k)}}{\Delta y}\right)\end{array}\right] \\ &= \hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{x,(i,j,k)}+\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{y,(i,j,k)}+\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{z,(i,j,k)}\end{aligned} \tag{9}$$

A square matrix can be formulated using the energy density component $\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{x,(i,j,k)}$.

$$\begin{aligned}\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{x,(i,j,k)} &= \vec{x}^T \overleftrightarrow{A}_{x,(i,j,k)}\vec{x} \\ &= \begin{pmatrix}H^{n+\frac{1}{2}}_{x,(i,j,k)} & E^{n+1}_{z,(i,j+1,k)} & E^{n+1}_{z,(i,j,k)} & E^{n+1}_{y,(i,j,k+1)} & E^{n+1}_{y,(i,j,k)}\end{pmatrix}\cdot\begin{pmatrix}\mu_x & \frac{-\Delta t}{2\Delta y} & \frac{+\Delta t}{2\Delta y} & \frac{+\Delta t}{2\Delta z} & \frac{-\Delta t}{2\Delta z} \\ \frac{-\Delta t}{2\Delta y} & \frac{1}{4}\varepsilon_z & 0 & 0 & 0 \\ \frac{+\Delta t}{2\Delta y} & 0 & \frac{1}{4}\varepsilon_z & 0 & 0 \\ \frac{+\Delta t}{2\Delta z} & 0 & 0 & \frac{1}{4}\varepsilon_y & 0 \\ \frac{-\Delta t}{2\Delta z} & 0 & 0 & 0 & \frac{1}{4}\varepsilon_y\end{pmatrix}\cdot\begin{pmatrix}H^{n+\frac{1}{2}}_{x,(i,j,k)} \\ E^{n+1}_{z,(i,j+1,k)} \\ E^{n+1}_{z,(i,j,k)} \\ E^{n+1}_{y,(i,j,k+1)} \\ E^{n+1}_{y,(i,j,k)}\end{pmatrix}\end{aligned}$$

The approach for $\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{y,(i,j,k)}$ and $\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{z,(i,j,k)}$ is analog. The positive definite property of the matrix $\overleftrightarrow{A}_{x,(i,j,k)}$, $\overleftrightarrow{A}_{y,(i,j,k)}$ and $\overleftrightarrow{A}_{z,(i,j,k)}$ is fulfilled if the smallest time-step is taken of the three derived conditions. The smallest chosen time-step protects the property that all determinants associated with all upper-left sub-matrices are positive definite. The derived stability condition is based on conservation of energy and is much stricter than the classical conditions based on conservation of proper wave propagation. It is important to mention that the split up into the energy densities $\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{x,(i,j,k)}$, $\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{y,(i,j,k)}$ and $\hat{w}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1}_{z,(i,j,k)}$ reflects the FDTD update scheme in cartesian coordinates and their successive build-up of their implicit poynting-vector.

$$0 < \Delta t < \min \left(\begin{array}{ccc} \sqrt{\dfrac{\mu_x}{\left(\frac{2}{\varepsilon_y}\right)\left(\frac{1}{\Delta y}\right)^2 + \left(\frac{2}{\varepsilon_z}\right)\left(\frac{1}{\Delta z}\right)^2}} & & \dfrac{1}{c\sqrt{2}\sqrt{\left(\frac{1}{\Delta y}\right)^2 + \left(\frac{1}{\Delta z}\right)^2}}, \\ \sqrt{\dfrac{\mu_y}{\left(\frac{2}{\varepsilon_z}\right)\left(\frac{1}{\Delta z}\right)^2 + \left(\frac{2}{\varepsilon_x}\right)\left(\frac{1}{\Delta x}\right)^2}} & \xrightarrow[\text{uniform grid}]{\text{isotropic material}} & \dfrac{1}{c\sqrt{2}\sqrt{\left(\frac{1}{\Delta z}\right)^2 + \left(\frac{1}{\Delta x}\right)^2}}, \\ \sqrt{\dfrac{\mu_z}{\left(\frac{2}{\varepsilon_x}\right)\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{2}{\varepsilon_y}\right)\left(\frac{1}{\Delta y}\right)^2}} & & \dfrac{1}{c\sqrt{2}\sqrt{\left(\frac{1}{\Delta x}\right)^2 + \left(\frac{1}{\Delta y}\right)^2}} \end{array} \right)$$

4. Wavelet-transformed FDTD

The first integral part of the wavelet-transformed FDTD scheme is the transformation of all spatial operators to achieve multi-resolution capabilities. The resulting property of the transformed operator implements automatically transformed field-components. For simpler and compacter notation for a 3D environment, component and direction indices are numbered in a modulo-three sense, e.g. $E_x = E_D$, $E_y = E_{D+1}$ and $E_z = E_{D-1}$. Starting with the Ampere's law of the classical FDTD update scheme and applying only a discrete wavelet-transformation in z-direction, we yield

$$\begin{aligned} E_D^{n+1} &= E_D^n + \frac{\Delta t}{\varepsilon_{r,D}} \left[\partial_{D+1} H_{D-1}^{n+\frac{1}{2}} - \partial_{D-1} H_{D+1}^{n+\frac{1}{2}} \right] \\ W_z \otimes E_D^{n+1} &= W_z \otimes \left[E_D^n + \frac{\Delta t}{\varepsilon_{r,D}} \left(\partial_{D+1} H_{D-1}^{n+\frac{1}{2}} - \partial_{D-1} H_{D+1}^{n+\frac{1}{2}} \right) \right] \\ \tilde{E}_D^{n+1} &= \tilde{E}_D^n + W_z \otimes \left[M_D \left(\partial_{D+1} H_{D-1}^{n+\frac{1}{2}} - \partial_{D-1} H_{D+1}^{n+\frac{1}{2}} \right) \right] \\ \tilde{E}_D^{n+1} &= \tilde{E}_D^n + W_z \otimes M_D \otimes W_z^{-1} \left[\partial_{D+1} W_z \otimes H_{D-1}^{n+\frac{1}{2}} - W_z \otimes \partial_{D-1} \otimes W_z^{-1} \otimes W_z \otimes H_{D+1}^{n+\frac{1}{2}} \right] \\ \tilde{E}_D^{n+1} &= \tilde{E}_D^n + \tilde{M}_D \left[\partial_{D+1} \tilde{H}_{D-1}^{n+\frac{1}{2}} - \tilde{\partial}_{D-1} \tilde{H}_{D+1}^{n+\frac{1}{2}} \right] \quad \text{with} \quad W_z \otimes W_z^{-1} = I \end{aligned} \tag{10}$$

The Faraday equation can be transformed straightforward. Since we apply a linear transformation and retain all field and operator information we inherit all advantages of the classical FDTD scheme, this includes the known stability condition. It may be pointed out, that the material operator comprised all three dimensions, whereas the derivative operator is only effective to its assigned direction. The incompatible direction of the derivative operator ∂_{D+1} keeps the structure untransformed. The transformed field is a spatially divided into an averaging $\mathrm{V}_{1,z}$ and a detail part $\mathrm{W}_{1,z}$. The transformed operators $\tilde{M}_D$ and $\tilde{\partial}_{D-1}$ simultaneously uses both segments and interlinks within the accessing structure those mentioned different resolutions. The thoughtless use of the transformed operators yields into longer numerical computation. This problem is tackled in the second integral part of the WT-FDTD scheme, which neglects unnecessary wavelet coefficients (field-components) by setting the specific operator coefficients to zero. The scheme gets numerical efficient, but the neglecting process is irreversible and makes the scheme nonlinear. The classical approach to derive a valid stability condition can not be applied.

5. Stability condition for the wavelet-transformed FDTD

The stability analysis for the wavelet-transformed FDTD needs a deeper view into the structure of the transformed field-components and operators. The complexity of those operators is highly dependent on the chosen filter coefficient, e.g. Haar, Daubechies or CDF filter. For simpler notation we will use the Haar filter and distinguish between the averaging ($\widehat{O}$, cap symbol) and detail segment ($\breve{O}$, bowl symbol) of the operator. The example uses one wavelet transformation into z-direction. The real energy density in the wavelet domain is now distributed on the averaging and detail segment. The outcomes of the distribution are six matrices and six terms of stability conditions. To be short and concise only one condition will be derived and the concluding global condition will be quoted without further explanation.

$$\tilde{M}_D = W_D \cdot M_D \cdot W_D^{-1} = \begin{pmatrix} \widehat{\overleftrightarrow{M}}_{\widehat{D}} & \widehat{\overleftrightarrow{M}}_{\breve{D}} \\ \breve{\overleftrightarrow{M}}_{\widehat{D}} & \breve{\overleftrightarrow{M}}_{\widehat{D}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{\Delta t(\varepsilon_{22}+\varepsilon_{11})}{2\varepsilon_{11}\varepsilon_{22}} & \cdots & 0 & \frac{\Delta t(\varepsilon_{22}-\varepsilon_{11})}{2\varepsilon_{11}\varepsilon_{22}} & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & \frac{\Delta t(\varepsilon_{n,n}+\varepsilon_{n-1,n-1})}{2\varepsilon_{n,n}\varepsilon_{n-1,n-1}} & 0 & \cdots & \frac{\Delta t(\varepsilon_{n,n}-\varepsilon_{n-1,n-1})}{2\varepsilon_{n,n}\varepsilon_{n-1,n-1}} \\ \frac{\Delta t(\varepsilon_{22}-\varepsilon_{11})}{2\varepsilon_{11}\varepsilon_{22}} & \cdots & 0 & \frac{\Delta t(\varepsilon_{22}+\varepsilon_{11})}{2\varepsilon_{11}\varepsilon_{22}} & \cdots & 0 \\ 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & \cdots & \frac{\Delta t(\varepsilon_{n,n}-\varepsilon_{n-1,n-1})}{2\varepsilon_{n,n}\varepsilon_{n-1,n-1}} & 0 & \cdots & \frac{\Delta t(\varepsilon_{n,n}+\varepsilon_{n-1,n-1})}{2\varepsilon_{n,n}\varepsilon_{n-1,n-1}} \end{pmatrix}$$

$$\tilde{\partial}_z = W_z \cdot \partial_z \cdot W_z^{-1} = \begin{pmatrix} \widehat{\overleftrightarrow{\partial}}_{\widehat{z}} & \widehat{\overleftrightarrow{\partial}}_{\breve{z}} \\ \breve{\overleftrightarrow{\partial}}_{\widehat{z}} & \widehat{\overleftrightarrow{\partial}}_{\breve{z}} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-1}{2\Delta z} & \frac{1}{2\Delta z} & \cdots & 0 & \frac{-1}{2\Delta z} & \frac{1}{2\Delta z} & \cdots & 0 \\ 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \frac{-1}{2\Delta z} & \frac{1}{2\Delta z} & 0 & \cdots & \frac{-1}{2\Delta z} & \frac{1}{2\Delta z} \\ 0 & \cdots & \cdots & \frac{-1}{2\Delta z} & 0 & \cdots & \cdots & \frac{-1}{2\Delta z} \\ \frac{1}{2\Delta z} & \frac{-1}{2\Delta z} & \cdots & 0 & \frac{-3}{2\Delta z} & \frac{-1}{2\Delta z} & \cdots & 0 \\ 0 & \ddots & \ddots & 0 & 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \frac{1}{2\Delta z} & \frac{-1}{2\Delta z} & 0 & \cdots & \frac{-3}{2\Delta z} & \frac{-1}{2\Delta z} \\ 0 & \cdots & \cdots & \frac{1}{2\Delta z} & 0 & \cdots & \cdots & \frac{-3}{2\Delta z} \end{pmatrix}$$

The short structure of the transformed material or derivative operator depicts the partitioning of the access and target points. Taking this into account, we yield

$$\begin{aligned}
\widehat{w}_{total,(i,j,k)}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1} = & \left[\begin{aligned} & \widehat{\mu}_{x,k\le\frac{N}{2}}\left(\widehat{H}_{x,(i,j,k)}^{n+\frac{1}{2}}\right)^2 + \frac{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{z,(i,j+1,k)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{z,(i,j,k)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}}{8}\left(\widehat{E}_{y,(i,j,k+1)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}}{8}\left(\widehat{E}_{y,(i,j,k)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{y,k>\frac{N}{2}}}{8}\left(\breve{E}_{y,(i,j,k+\frac{N}{2}+2)}^{n+1}\right)^2 + \\ & + \frac{\widehat{\varepsilon}_{y,k>\frac{N}{2}}}{8}\left(\breve{E}_{y,(i,j,k+\frac{N}{2}+1)}^{n+1}\right)^2 - \Delta t \cdot \widehat{H}_{x,(i,j,k)}^{n+\frac{1}{2}}\left(\frac{\widehat{E}_{z,(i,j+1,k)}^{n+1} - \widehat{E}_{z,(i,j,k)}^{n+1}}{\Delta y} - \left(\frac{\widehat{E}_{y,(i,j,k+1)}^{n+1} - \widehat{E}_{y,(i,j,k)}^{n+1}}{2\Delta z} + \frac{\breve{E}_{y,(i,j,k+\frac{N}{2}+2)}^{n+1} - \breve{E}_{y,(i,j,k+\frac{N}{2}+1)}^{n+1}}{2\Delta z}\right)\right) \end{aligned} \right] + \\
& + \left[\begin{aligned} & \widehat{\mu}_{y,k\le\frac{N}{2}}\left(\widehat{H}_{y,(i,j,k)}^{n+\frac{1}{2}}\right)^2 + \frac{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}}{8}\left(\widehat{E}_{x,(i,j,k+1)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}}{8}\left(\widehat{E}_{x,(i,j,k)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{x,k>\frac{N}{2}}}{8}\left(\breve{E}_{x,(i,j,k+\frac{N}{2}+2)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{x,k>\frac{N}{2}}}{8}\left(\breve{E}_{x,(i,j,k+\frac{N}{2}+1)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{z,(i+1,j,k)}^{n+1}\right)^2 + \\ & \frac{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{z,(i,j,k)}^{n+1}\right)^2 - \Delta t \cdot \widehat{H}_{y,(i,j,k)}^{n+\frac{1}{2}}\left(\left(\frac{\widehat{E}_{x,(i,j,k+1)}^{n+1} - \widehat{E}_{x,(i,j,k)}^{n+1}}{2\Delta z} + \frac{\breve{E}_{x,(i,j,k+\frac{N}{2}+2)}^{n+1} - \breve{E}_{x,(i,j,k+\frac{N}{2}+1)}^{n+1}}{2\Delta z}\right) - \frac{\widehat{E}_{z,(i+1,j,k)}^{n+1} - \widehat{E}_{z,(i,j,k)}^{n+1}}{\Delta x}\right) \end{aligned} \right] + \\
& + \left[\begin{aligned} & \widehat{\mu}_{z,k\le\frac{N}{2}}\left(\widehat{H}_{z,(i,j,k)}^{n+\frac{1}{2}}\right)^2 + \frac{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{y,(i+1,j,k)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{y,(i,j,k)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{x,(i,j+1,k)}^{n+1}\right)^2 + \frac{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}}{4}\left(\widehat{E}_{x,(i,j,k)}^{n+1}\right)^2 - \\ & -\Delta t \cdot \widehat{H}_{z,(i,j,k)}^{n+\frac{1}{2}}\left(\frac{\widehat{E}_{y,(i+1,j,k)}^{n+1} - \widehat{E}_{y,(i,j,k)}^{n+1}}{\Delta x} - \frac{\widehat{E}_{x,(i,j+1,k)}^{n+1} - \widehat{E}_{x,(i,j,k)}^{n+1}}{\Delta y}\right) \end{aligned} \right] \\
= & \ \widehat{w}_{x,(i,j,k)}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1} + \widehat{w}_{y,(i,j,k)}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1} + \widehat{w}_{z,(i,j,k)}^{n+\frac{1}{2}\le\frac{t}{\Delta t}\le n+1} \qquad (11)
\end{aligned}$$

The detail coefficients hold the detail part of the energy density and the neglecting strategy destroys their assigned energy density. This could cause an unphysical scene, but during the fact that most of the electromagnetic field is smooth and continuous the resulting differences are small to be omitted. Conservation of energy is maintained if the neglecting strategy is proper and reliable. We developed two reliable strategies yielding higher numerical efficiency in comparison to the conventional FDTD scheme which are not part of this publication. The square matrices are generated as noted before and their assigned positive definite property is maintained if the smallest time-step Δt is chosen.

$$
0 < \Delta t < \min \left(\begin{array}{l}
\widehat{\Delta t}_{yz} \ni \left[\sqrt{\dfrac{\widehat{\mu}_{x,k\le\frac{N}{2}}}{\left(\frac{1}{\Delta z}\right)^2 \left(\frac{1}{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}} + \frac{1}{\widehat{\varepsilon}_{y,k>\frac{N}{2}}}\right) + \left(\frac{1}{\Delta y}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}\right)}} \xrightarrow[\text{detail coefficients}]{\text{with neglected}} \sqrt{\dfrac{\widehat{\mu}_{x,k\le\frac{N}{2}}}{\left(\frac{1}{\Delta z}\right)^2 \left(\frac{1}{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}}\right) + \left(\frac{1}{\Delta y}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}\right)}} \right], \\
\widehat{\Delta t}_{xz} \ni \left[\sqrt{\dfrac{\widehat{\mu}_{y,k\le\frac{N}{2}}}{\left(\frac{1}{\Delta x}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}\right) + \left(\frac{1}{\Delta z}\right)^2 \left(\frac{1}{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}} + \frac{1}{\widehat{\varepsilon}_{x,k>\frac{N}{2}}}\right)}} \xrightarrow[\text{detail coefficients}]{\text{with neglected}} \sqrt{\dfrac{\widehat{\mu}_{y,k\le\frac{N}{2}}}{\left(\frac{1}{\Delta x}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{z,k\le\frac{N}{2}}}\right) + \left(\frac{1}{\Delta z}\right)^2 \left(\frac{1}{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}}\right)}} \right], \\
\widehat{\Delta t}_{xy} \ni \left[\sqrt{\dfrac{\widehat{\mu}_{z,k\le\frac{N}{2}}}{\left(\frac{1}{\Delta x}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}}\right) + \left(\frac{1}{\Delta y}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}}\right)}} \xrightarrow[\text{detail coefficients}]{\text{with neglected}} \sqrt{\dfrac{\widehat{\mu}_{z,k\le\frac{N}{2}}}{\left(\frac{1}{\Delta x}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{y,k\le\frac{N}{2}}}\right) + \left(\frac{1}{\Delta y}\right)^2 \left(\frac{2}{\widehat{\varepsilon}_{x,k\le\frac{N}{2}}}\right)}} \right], \\
\breve{\Delta t}_{yz} \ni \left[\sqrt{\dfrac{\breve{\mu}_{x,k>\frac{N}{2}}}{\left(\frac{1}{\Delta z}\right)^2 \left(\frac{1}{\breve{\varepsilon}_{y,k>\frac{N}{2}}} + \frac{5}{\breve{\varepsilon}_{y,k>\frac{N}{2}}}\right) + \left(\frac{1}{\Delta y}\right)^2 \left(\frac{2}{\breve{\varepsilon}_{z,k>\frac{N}{2}}}\right)}} \xrightarrow[\text{detail coefficients}]{\text{with neglected}} \text{undefined}, \vec{x}^T \breve{\overleftrightarrow{w}}_{x,(i,j,k)} \vec{x} = 0 \right], \\
\breve{\Delta t}_{xz} \ni \left[\sqrt{\dfrac{\breve{\mu}_{y,k>\frac{N}{2}}}{\left(\frac{1}{\Delta x}\right)^2 \left(\frac{2}{\breve{\varepsilon}_{z,k>\frac{N}{2}}}\right) + \left(\frac{1}{\Delta z}\right)^2 \left(\frac{1}{\breve{\varepsilon}_{x,k>\frac{N}{2}}} + \frac{5}{\breve{\varepsilon}_{x,k>\frac{N}{2}}}\right)}} \xrightarrow[\text{detail coefficients}]{\text{with neglected}} \text{undefined}, \vec{x}^T \breve{\overleftrightarrow{w}}_{y,(i,j,k)} \vec{x} = 0 \right], \\
\breve{\Delta t}_{xy} \ni \left[\sqrt{\dfrac{\breve{\mu}_{z,k>\frac{N}{2}}}{\left(\frac{1}{\Delta x}\right)^2 \left(\frac{2}{\breve{\varepsilon}_{y,k>\frac{N}{2}}}\right) + \left(\frac{1}{\Delta y}\right)^2 \left(\frac{2}{\breve{\varepsilon}_{x,k>\frac{N}{2}}}\right)}} \xrightarrow[\text{detail coefficients}]{\text{with neglected}} \text{undefined}, \vec{x}^T \breve{\overleftrightarrow{w}}_{z,(i,j,k)} \vec{x} = 0 \right]
\end{array} \right)
$$

The solutions of the stability condition are asymmetrical, since we transformed the system only in one direction. The derived condition is similar to the classical one, except of the transformed elements. Simulations using the wavelet-transformed FDTD scheme show stability for those conditions and beyond them until the classical courant limit is reached. Even if the detail coefficients are neglected stability is maintained at the courant limit. This is in fact surprising, but on the knowledge of the author the presented approach is the first published analytical formulation for the stability condition on spatial wavelet-transformed FDTD schemes without using the norm on operators.

6. Conclusions

In this paper we have presented the energy-based stability condition firstly presented by Kung. Shortly introduced the spatial wavelet-transformed FDTD method and applied the stability analysis on the method with one transformation direction. The derived stability condition was confirmed through simulations, but the test also show the possibility to enlarge the condition until the classical time limit is reached. We conclude the energy-based condition as a "strong condition" which could be analytically derived. At the moment the results of classical stability analysis is labeled as a "weak condition" since the neglecting strategies on operator coefficients with multi-resolution capabilities could not be included.

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